Date: July 16, 2019 **Name:** John Chillery

PhD Preliminary Examination in Analysis

Department of Mathematics

New Mexico Tech

2019

1. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function on $[0,1]$. Suppose that $f(0) = 0$, and there is a real number α such that $|f'(x)| \le \alpha |f(x)|$ for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on $[0,1]$ defined by $f_n(x) = \frac{1}{1+n^2x^2}$. Does the sequence $\{f_n\}$ uniformly converge on $[0,1]$? Justify your answer.

3. Let $\alpha, x_1 \in \mathbb{R}_+$ be positive real numbers such that $x_1 > \alpha$. Let $(x_n)_{n=1}^{\infty}$ be a sequence defined recursively by
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha^2}{x_n}\right), \qquad n \ge 1\,.$$
Prove that the sequence $(x_n)_{n=1}^{\infty}$ is convergent and find its limit.

4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$. Suppose that there exists $\alpha \in (0,1)$ such that for every $x \in [0,1]$ there exists $z \in [0,1]$ such that $|f(z)| \le \alpha |f(x)|$. Prove that there exists a point $c \in [0,1]$ such that $f(c) = 0$.

5. Let K be a positive real number and $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)| \le K|z|$ for all $z \in \mathbb{C}$. Show that there is a constant a such that $f(z) = az$ for all $z \in \mathbb{C}$.

6. Let P be a polynomial. Let C be a sufficiently large circle centered at the origin and oriented counterclockwise. Show that the integral
$$I = \oint_C \frac{dz}{2\pi i} z \frac{P'(z)}{P(z)}$$
is equal to the sum of the roots of the polynomial P counted with multiplicities.

7. Let f and g be analytic functions on a region A in the complex plane. Suppose that f is injective (one-to-one) and $f'(z) \ne 0$ for any $z \in A$. Let C be a simple (without self-intersections) closed contour in A oriented counterclockwise. Let z_0 be a point inside C. Show that
$$\oint_C \frac{dz}{2\pi i}\, \frac{g(z)}{f(z) - f(z_0)} = \frac{g(z_0)}{f'(z_0)}\,.$$

8. Let $n \in \mathbb{Z}_+$ be a positive integer and C be the unit circle in the complex plane centered at the origin oriented counterclockwise. Evaluate the integral
$$\oint_C \left(z + \frac{1}{z}\right)^n \frac{dz}{z}\,.$$