Probability and Statistics, Sample Prelim Questions, Spring 2021

1. Let $X_1, X_2, \ldots, X_n$ be an independent random sample drawn from a Poisson distribution with mean λ, and $\lambda > 0$ is an unknown parameter.

$$f(x|\lambda) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!} & \text{for } x = 0, 1, 2\ldots \\ 0 & \text{elsewhere} \end{cases}$$

Consider two estimators for λ, $T_1 = \frac{1}{n}\sum_{i=1}^n X_i$ and $T_2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

(a) Are T_1 and T_2 unbiased estimators for λ? Justify your answer.

(b) Which estimator is more efficient for λ, T_1 or T_2? Justify your answer.

(c) Calculate the Cramer-Rao lower bound for unbiased estimator of λ^2.

(d) Suppose that λ has an exponential prior distribution with mean $\theta > 0$,

$$f(\lambda|\theta) = \begin{cases} \frac{1}{\theta}e^{-\frac{\lambda}{\theta}} & \text{for } \lambda > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Derive the posterior distribution of λ.

(e) Compute the posterior mean of λ and show that the posterior mean of λ is consistent for λ as $n \to \infty$.

2. Consider the following joint density for random variables X and Y:

$$f(x, y) = \begin{cases} k\,(1-y), & \text{for } 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find k.

(b) Evaluate $\mathbb{E}\,(X)$ and $\mathbb{E}\,(X^2)$. Find the variance of X.

(c) Derive the conditional density $f(y|x)$ and the conditional expectation, $\mathbb{E}\,[1-Y \mid X]$. Hence or otherwise, evaluate $\mathbb{E}\,(Y)$ and $Cov(X, Y)$.

(d) Find $P(Y < 2X)$.

3. Suppose that $X_1, X_2, \ldots, X_n$ is an i.i.d. sample from a normal distribution with mean μ and variance 1. Remember that the density of X_i is $\frac{1}{\sqrt{2\pi}}\exp(-\frac{(x-\mu)^2}{2})$.

(a) Show that the likelihood ratio test of the null hypothesis $\mu = 3$ against the alternative hypothesis $\mu = 5$ rejects the null hypothesis if and only if $\bar{X} = \frac{\sum_{i=1}^n X_i}{n} > c$ for some c.

(b) Find the value of c if $n = 6$ and the probability of Type I error is 0.05.

4. Consider the linear regression model

$$y = X\beta + \epsilon$$

where X is a matrix of size m by n and rank n, y is a vector of length m, β is a vector of length n, and ϵ is a random vector with the multivariate normal distribution $N(0, \sigma^2 I)$.

(a) Write down the likelihood function.

(b) Use the likelihood function from part (a) to Show that the MLE is

$$\hat{\beta} = \arg\min \|X\beta - y\|_2^2$$

5. Consider the classical Gambler's Ruin Problem. Me and my friend are tossing a coin, if the coin comes up Heads, I win \$1 from my friend, if Tails, I lose \$1. I start with a dollars and my friend starts with b dollars. The game ends when one of us loses all their money. We will model the amount of money I have after ith toss as a Markov Chain X_i, for a nonegative integer i, and $X_0 = a$. The state space is $\{0, 1, ..., N\}$, where $N = a + b$.

(a) Specify the transition matrix.

(b) Classify all the states of the Markov Chain as recurrent or transient, with explanations.

(c) Argue that eventually the game will end with probability 1.

(d) Let u_i be the probability that, starting at the state i, I will eventually win the game. Set up a system of equations describing $u_i, i = 0, 1, ..., N$.

(e) Solve the system.

6. Consider a Poisson process $X(t)$ with intensity λ, so that

$$p_k(t) = P(X(t) = k) = \exp(-\lambda t)\frac{(\lambda t)^k}{k!}.$$

Let W_k be the time when kth event happens.

(a) Using the formula for $p_k(t)$, derive an expression for the density of W_k.

(b) Calculate $\mathbb{E}\,(W_3 \,|\, X(t) = 5)$ and $\mathbb{E}\,(W_5 \,|\, X(t) = 3)$

7. Consider the single-server queue with i.i.d. Exponential (with the mean α) interarrival and i.i.d. Exponential (with the mean β) service times. Let $X(t)$ be the number of total customers (both under service and in queue) in the system at the time t. Model $X(t)$ as a life-and-death process. Find the limiting distribution $\pi_k = \lim_{t\to\infty} P(X(t) = k)$ and the conditions under which it exists.