DE Preliminary Exam Summer 2018

Do all the problems on your own and show all your work for full credit.

ODES

1. Consider the system with $r^2 = x^2 + y^2$

$$\begin{aligned} \dot{x} =& -y + x(r^4 - 3r^2 + 1), \\ \dot{y} =& x + y(r^4 - 3r^2 + 1). \end{aligned}$$

Show that the origin is an unstable focus for this system and use the Poincare-Bendixon Theorem to show that there is a periodic orbit in the annular region

$$A = \{\mathbf{x} \in \mathbf{R^2} \mid \mathbf{0} < |\mathbf{x}| < \mathbf{1}\}.$$

2. For the system

$$\begin{aligned} \dot{x} =& -3(x+y)^2 - 2x, \\ \dot{y} =& -3(x+y)^2. \end{aligned}$$

   (i) Find the critical point(s).

   (ii) What does the linearization tell you?

   (iii) This is a gradient system. What does the Liapunov function tell you?

3. Analyze

$$x_{n+1} = \frac{a x_n^2}{1 + x_n^2}.$$

PDES

1. Determine the solution of the IVP (unidirectional, nonlinear wave equation):

$$u_t + u u_x = 0, \quad u(x,0) = \begin{cases} 2, \ x \in (-\infty, 0], \\ \frac{5-x}{3}, \ x \in (0, 1], \\ 1, \ x \in (1, \infty). \end{cases}$$

2. Find the solution of the damped wave equation

$$\begin{aligned}
&u_{tt} + 2\beta u_t = a^2 u_{xx},\ 0 < x < c,\\
&u(0,t) = 0,\ u_x(c,t) = g(t)\ (\textit{boundary conditions});\\
&u(x,0) = u_t(x,0) = 0,\ 0 < x < c\ (\textit{initial conditions}).
\end{aligned}$$

You may assume that $a > 0,\ 0 < \beta < \frac{a\pi}{2c},\ g(0) = 0$ and $g'(0) = 0$.

3. (a) Find the solution of the BVP

$$2u_{xx} + 2u_{xy} + u_{yy} = 0, \quad -\infty < x < \infty,\ 0 < y < \infty$$

with the boundary conditions

$$u(x,0) = f(x),\ u < \infty \text{ as } x^2 + y^2 \to \infty.$$

(b) Determine the closed form (simplified) solution for the specific case when

$$f(x) = \begin{cases} T_0 \text{ (a constant)}, & x \in [-1,1], \\ 0,\ x \notin [-1,1]. \end{cases}$$