DE Preliminary Exam Fall 2017

Do all the problems on your own and show all your work for full credit.

1. Show that the planar system

$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$

has a periodic orbit in the annular region $A = \left\{ \mathbf{x} \in \mathbf{R}^2 | \quad 1 < |\mathbf{x}| < \sqrt{2} \right\}$.

2. Determine the flow $\phi_t : \ \mathbf{R}^3 \to \mathbf{R}^3$ for the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ with } \mathbf{f}(\mathbf{x}) = (-x_1, -x_2 + x_1^2, x_3 + x_1^2)^T.$$

Show also that the set $S = \left\{ \mathbf{x} \in \mathbf{R}^3 : \quad x_3 = -\dfrac{x_1^2}{3} \right\}$ is invariant under the flow ϕ_t.

3. Analyze

$$x_{n+1} = \frac{a x_n^2}{1 + x_n^3} \text{ for } a > 0.$$

4. Determine the solution of the IVP (unidirectional, nonlinear wave equation):

$$u_t + u u_x = 0, \quad u(x,0) = \begin{cases} 2, \ x \in (-\infty, 0], \\ \frac{5-x}{3}, \ x \in (0,1], \\ 1, \ x \in (1, \infty). \end{cases}$$

5. Find the solution of the damped wave equation

$$\begin{aligned} &u_{tt} + 2\beta u_t = a^2 u_{xx}, \ 0 < x < c, \\ &u(0,t) = 0, \ u_x(c,t) = g(t) \ (\textit{boundary conditions}); \\ &u(x,0) = u_t(x,0) = 0, \ 0 < x < c \ (\textit{initial conditions}). \end{aligned}$$

You may assume that $a > 0, \ 0 < \beta < \frac{a\pi}{2c}, \ g(0) = 0$ and $g'(0) = 0$.

6. (a) Find the solution of the BVP

$$2u_{xx} + 2u_{xy} + u_{yy} = 0, \quad -\infty < x < \infty, \ 0 < y < \infty$$

with the boundary conditions

$$u(x,0) = f(x), \ u < \infty \text{ as } x^2 + y^2 \to \infty.$$

(b) Determine the closed form (simplified) solution for the specific case when

$$f(x) = \begin{cases} T_0 \text{ (a constant)}, \ x \in [-1,1], \\ 0, \ x \notin [-1,1]. \end{cases}$$