Preliminary Exam in Differential Equations- Aug. 2024

There are two parts for the exam. Part I is related to Ordinary Differential Equations, while the second part is related to Partial Differential Equations! You have 24 hours to complete the exam. You must work on the problems on your own, but you may use textbooks, lecture notes, Matlab and Maple. Online resources and help from other people are not allowed. Show all your work for full credits.

Ordinary Differential Equation Part

1. Show that the set $S = \{x \in \mathbf{R}^3 | x_2 = -\frac{x_1^2}{4}\}$ is invariant under the flow $\phi_t : \mathbf{R}^2 \to \mathbf{R}^2$ for the nonlinear system
$$\dot{\mathbf{x}} = f(\mathbf{x}), \text{ where } f(\mathbf{x}) = \begin{bmatrix} -x_1 \\ 2x_2 + x_1^2 \end{bmatrix}$$

2. Consider the nonlinear system
$$\dot{x}_1 = -\, x_1, \quad \dot{x}_2 = -x_2 + x_1^2, \quad \dot{x}_3 = x_3 + x_1^2.$$
Find its stable and unstable manifolds at its unique equilibrium point.

3. Use Poincare-Bendixson Theorem to show that the following planar system
$$\dot{x} = x - y - x^3, \quad \dot{y} = x + y - y^3$$
has a periodic orbit in the annular region $A = \{\mathbf{x} \in \mathbf{R}^2 | 1 < |\mathbf{x}| < \sqrt{2}\}$. (*For this problem, rigorous mathematical analysis is required. Phase portrait obtained numerically will not be enough!*)

Partial Differential Equation Part

1. Determine the solution of the initial value problem (unidirectional, nonlinear wave equation, with *speed* $c = 2u$):
$$u_t + 2uu_x = 0 \qquad , \qquad u(x,0) = \begin{cases} 1 & , \quad -\infty < x < 0 \\ \frac{1}{2}x & , \quad 0 \le x \le \frac{52}{27} \\ 0 & , \quad \frac{52}{27} < x < \infty \end{cases}$$
In particular, determine when the wave breaks **and** the equation of the shock(s). Summarize your final solution in the form of an x-t plane graph, indicating: $u = u_1(x,t)$ in Region$_1$, $u = u_2(x,t)$ in Region$_2$, etc.

2. Two parts for the following problem.

   (a) Find the solution of the BVP

$$u_{xx} + 4u_{yy} = 0 \quad , \quad 0 < x < \infty \quad , \quad 0 < y < \infty$$

   with the boundary conditions $u(x,0) = f(x)$, $u(0,y) = 0$, $u < \infty$ as $x^2 + y^2 \to \infty$.

   (b) Determine the closed form (simplified) solution for the specific case when

$$f(x) = \begin{cases} T_0 \text{ (a constant)} & , \quad x \in [1,2] \\ 0 & , \quad x \notin [1,2] \end{cases}$$

3. Find the solution of the BVP consisting of the (damped wave equation) PDE

$$u_{tt} + 2u_t = a^2 u_{xx} \ , \quad 0 < x < c$$

   subject to the boundary conditions:

$$u(0,t) = 0 \quad , \quad u(c,t) = g(t)$$

   and the initial conditions

$$u(x,0) = u_t(x,0) = 0 \ , \quad 0 < x < c$$

   You may assume that $a > 0$, $a\pi > c$, $g(0) = 0$ and $g'(0) = 0$.